

General class of wormhole geometries in conformal Weyl gravity

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In this work, a general class of wormhole geometries in conformal Weyl gravity is analyzed. A wide variety of exact solutions of asymptotically flat spacetimes is found, in which the stress energy tensor profile differs radically from its general relativistic counterpart. In particular, a class of geometries is constructed that satisfies the energy conditions in the throat neighborhood, which is in clear contrast to the general relativistic solutions.

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I. INTRODUCTION

The Einstein field equation reflects the dynamics of general relativity, and is formally obtained from the Einstein-Hilbert action, $I_{EH} = \int d^4x \sqrt{-g} R$, where R is the curvature scalar. However, the latter action can be generalized to include other scalar invariants. An intriguing example is conformal Weyl gravity, involving the following purely gravitational sector of the action

$$I_W = -\alpha \int d^4x \sqrt{-g} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta}, \quad (1)$$

where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor, and α is a dimensionless gravitational coupling constant. It was argued in Ref. [1, 2] that in analogy to the principle of local gauge invariance that severely restricts the structure of possible Lorentz invariant actions in flat spacetimes, then the principle of local conformal invariance is a requisite invariance principle in curved spacetimes. The latter principle requires that the gravitational action to remain invariant under the conformal transformations $g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x)$. The conformal Weyl tensor

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - g_{\mu[\alpha}R_{\beta]\nu} + g_{\nu[\alpha}R_{\beta]\mu} + \frac{1}{3}Rg_{\mu[\alpha}g_{\beta]\nu}, \quad (2)$$

also transforms as $C_{\mu\nu\alpha\beta} \rightarrow \Omega^2(x)C_{\mu\nu\alpha\beta}$. The action I_W is an interesting theoretical construct, for instance, being a strictly conformally invariant theory, particle masses may possibly arise through the spontaneous symmetry breaking of the action [1].

Being a fourth order gravity theory, with respect to the derivatives of the metric, finding exact solutions of the gravitational field equations yields a formidable endeavor. Nevertheless, the exact vacuum exterior solution for a static and spherically symmetric spacetime in locally conformal invariant Weyl gravity was found in Ref. [1]. The solution contains the exterior Schwarzschild solution and provides a potential explanation for observed

galactic rotation curves without the need for dark matter [3, 4]. The time-dependent spherically symmetric solution was further explored in Ref. [2]. The exact solutions to the Reissner-Nordström problem associated with a static and spherically symmetric point electric and/or magnetic charge coupled to fourth-order conformal Weyl gravity were found [5]. In addition to this, exact solutions associated with the fourth-order Kerr and Kerr-Newman problems in which a stationary and axially symmetric rotating system with or without electric and/or magnetic charge is coupled to gravity, were further explored [5]. The causal structure, using Penrose diagrams, of the static spherically symmetric vacuum solution to conformal Weyl gravity was also investigated [6]. New vacuum solutions were found using a covariant $(2+2)$ -decomposition of the field equation, which covers the spherically and the plane symmetric space-times as special subcases [7]. Exact topological black hole solutions of conformal Weyl gravity, with negative, zero or positive scalar curvature at infinity were also found [8], the former generalizing the well-known topological black holes in anti-de Sitter gravity.

The weak-field limit of conformal Weyl gravity for an arbitrary spherically symmetric static distribution of matter in the physical gauge with a constant scalar field was also analyzed [9], and it was argued that the conformal theory of gravity is inconsistent with the Solar System observational data. In a cosmological context, exact analytical solutions to conformal Weyl gravity for the matter and radiation dominated eras, and the primordial nucleosynthesis process were exhaustively analyzed. It was found that the cosmological models are unlikely to reproduce the observational properties of our Universe, as they fail to fulfill the observational constraints on present cosmological parameters and on primordial light element abundances [10]. In Ref. [11] it was also argued that in the limit of weak fields and non-relativistic velocities the theory does not agree with the predictions of general relativity, and is therefore ruled out by Solar System observations. Nevertheless, in Ref. [12], it was counter-argued in the presence of macroscopic long range scalar fields, the standard Schwarzschild phenomenology is still recovered. To check the viability of Weyl grav-

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ity, two additional classical tests of the theory, namely, the deflection of light and time delay in the exterior of a static spherically symmetric source were analyzed, and it was shown that the parameters fit the experimental constraints [13, 14].

An interesting application of conformal Weyl gravity would be to analyze traversable wormhole solutions in the theory. We emphasize that an important and intriguing challenge in wormhole physics is the quest to find a realistic matter source that will support these exotic spacetimes. In classical general relativity, wormholes are supported by exotic matter, which involves a stress energy tensor that violates the null energy condition (NEC) [15, 16]. Note that the NEC is given by $T_{\mu\nu}k^\mu k^\nu \geq 0$, where k^μ is *any* null vector. Several candidates have been proposed in the literature, amongst which we refer to solutions in higher dimensions, for instance in Einstein-Gauss-Bonnet theory [17, 18], wormholes on the brane [19, 20]; solutions in Brans-Dicke theory [21]; wormhole solutions in semi-classical gravity (see Ref. [22] and references therein); exact wormhole solutions using a more systematic geometric approach were found [23]; and solutions supported by equations of state responsible for the cosmic acceleration [24], etc (see Refs. [25, 26] for more details and [26] for a recent review). In conformal Weyl gravity, as the gravitational field equations differ radically from the Einstein field equation, one would expect a wider class of solutions. This is indeed the case, and the solutions found contain interesting physical properties and characteristics, amongst which we refer to a zero or positive radial pressure at the throat, or more important the non-violation of the energy conditions in the throat neighborhood, contrary to their general relativistic counterparts.

This paper is outlined in the following manner: In Section II, we outline the general formalism and the gravitational field equations governing static and spherically symmetric spacetimes in conformal Weyl gravity. In Section III, we further explore specific wormhole solutions, and finally, in Section IV, we conclude.

II. GENERAL FORMALISM

A. Gravitational field equations

The metric used throughout this work, in curvature coordinates, is given by

$$ds^2 = -B(r) dt^2 + A(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (3)$$

The Weyl action, Eq. (1), may be simplified by noting that the quantity

$$\sqrt{-g} (R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} R^{\mu\nu} + R^2), \quad (4)$$

is a total divergence, and thus I_W may be rewritten as

$$I_W = -2\alpha \int d^4x \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right). \quad (5)$$

Varying the action with respect to the metric $g_{\mu\nu}$ provides the following relationship

$$(-g)^{-1/2} g_{\mu\alpha} g_{\nu\beta} \frac{\delta I_W}{\delta g_{\alpha\beta}} = -2\alpha \left[W_{\mu\nu}^{(2)} - \frac{1}{3} W_{\mu\nu}^{(1)} \right], \quad (6)$$

with $W_{\mu\nu}^{(1)}$ and $W_{\mu\nu}^{(2)}$ given by

$$W_{\mu\nu}^{(1)} = 2g_{\mu\nu} R^{;\beta}_{;\beta} - 2R_{;\mu\nu} - 2RR_{\mu\nu} + \frac{1}{2}g_{\mu\nu} R^2, \quad (7)$$

and

$$W_{\mu\nu}^{(2)} = \frac{1}{2}g_{\mu\nu} R^{;\beta}_{;\beta} + R_{\mu\nu}^{;\beta}_{;\beta} - R_{\mu}^{\beta}{}_{;\nu\beta} - R_{\nu}^{\beta}{}_{;\mu\beta} - 2R_{\mu\beta} R_{\nu}^{\beta} + \frac{1}{2}g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta}, \quad (8)$$

respectively.

The stress energy tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta(g^{\mu\nu})}, \quad (9)$$

where L_m is the Lagrangian density corresponding to matter.

The final gravitational field equation is given by

$$4\alpha W_{\mu\nu} = T_{\mu\nu}, \quad (10)$$

with $W_{\mu\nu} = W_{\mu\nu}^{(2)} - \frac{1}{3}W_{\mu\nu}^{(1)}$. Both sides are symmetric, traceless and covariantly conserved. Note that the intrinsic Newtonian constant that arises in the Einstein-Hilbert action is absent.

Determining $W_{\mu\nu}$ from Eqs. (7) and (8) presents a formidable endeavor. However, one may use the fact that for an arbitrary action $I = \int \sqrt{-g} d^4x L$, and using the metric (3), the term W^{rr} may be deduced from [1]

$$\begin{aligned} \sqrt{-g} W^{rr} &= -\frac{1}{2\alpha} \frac{\delta I}{\delta A} = \frac{\partial}{\partial A} (\sqrt{-g} L) - \frac{\partial}{\partial r} \left(\sqrt{-g} \frac{\partial L}{\partial A'} \right) \\ &\quad + \frac{\partial^2}{\partial r^2} \left(\sqrt{-g} \frac{\partial L}{\partial A''} \right), \end{aligned} \quad (11)$$

where the prime denotes a derivative with respect to the radial coordinate, r . Likewise for W^{tt} from $\delta I/\delta B$, etc. However, rather than use this method, which for calculational purposes is rather intractable, the gravitational tensor components W^{tt} and $W^{\theta\theta}$ may be determined from the Bianchi and trace identities, and given in terms of W^{rr} [1].

From the Bianchi identity,

$$W^{\mu\nu}{}_{;\mu} = (-g)^{-1/2} \left[(-g)^{1/2} W^{\mu\nu} \right]_{;\mu} + \Gamma^{\nu}_{\mu\lambda} W^{\mu\lambda} = 0, \quad (12)$$

one obtains the following relationship

$$\mathcal{D} W^{rr} + \frac{B'}{2A} W^{tt} - \frac{2r}{A} W^{\theta\theta} = 0, \quad (13)$$

where we have defined

$$\mathcal{D} = \frac{\partial}{\partial r} + \frac{2}{r} + \frac{A'}{A} + \frac{B'}{2B}, \quad (14)$$

for notational simplicity.

From the trace identity, $W^\mu{}_\mu = 0$, one obtains the following relationship

$$-B W^{tt} + A W^{rr} + 2r^2 W^{\theta\theta} = 0. \quad (15)$$

Finally, using Eqs. (13) and (15), the gravitational tensor components W^{tt} and $W^{\theta\theta}$ are related to W^{rr} through the following expressions

$$W^{tt} = \frac{A}{B - B'r/2} (1 + r \mathcal{D}) W^{rr}, \quad (16)$$

$$W^{\theta\theta} = \frac{A}{4r(B - B'r/2)} (B' + 2B \mathcal{D}) W^{rr}, \quad (17)$$

so that all the information is contained in the W^{rr} term.

The stress-energy tensor components, through the gravitational field equation, are given by

$$\rho = -4\alpha W^t{}_t, \quad p_r = 4\alpha W^r{}_r, \quad p_t = 4\alpha W^\theta{}_\theta, \quad (18)$$

in which $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure, and $p_t(r)$ is the lateral pressure measured in the orthogonal direction to the radial direction. Note that in conformal Weyl gravity, the stress energy tensor components are constrained through the trace identity, i.e., $-\rho + p_r + 2p_t = 0$.

Although extremely lengthy, we present the relevant gravitational terms, namely, $W^r{}_r$ and $W^t{}_t$, which will be used extensively throughout this work:

$$\begin{aligned} W^r{}_r = & \left\{ \left[4A^2 B^2 (2B' B''' - B''^2) - 4ABB'' (3AB'^2 + 2A' BB') + 7A^2 B'^4 + 6AA' BB'^3 + B^2 B'^2 (7A'^2 - 4AA'') \right] r^4 \right. \\ & + \left[-16A^2 B^3 B''' + 16AB^2 B'' (3AB' + A'B) - 20A^2 BB'^3 - 16AA' B^2 B'^2 + 4B^3 B' (4AA'' - 7A'^2) \right] r^3 \\ & + \left[-4A^2 B^2 (8BB'' + B'^2) + 8AA' B^3 B' + 4B^4 (7A'^2 - 4AA'') \right] r^2 \\ & \left. + 32A^2 B^3 B'r + 16A^2 B^4 (A^2 - 1) \right\} / (48A^4 B^4 r^4), \end{aligned} \quad (19)$$

and

$$\begin{aligned} W^t{}_t = & \left\{ \left[16A^2 B^3 B'''' - 48AB^2 (AB' + A'B) B''' - 36A^2 B^2 (B'')^2 + 4ABB' (29AB' + 27A'B) B'' \right. \right. \\ & - 4B^3 B'' (8AA'' - 19A'^2) - 49A^2 B'^4 - 58AA' BB'^3 + (24AA'' - 57A'^2) B^2 B'^2 \\ & + 4 \left(-2AA''' + 13A'A'' - 14 \frac{A'^2}{A} \right) B^3 B' \left. \right] r^4 + \left[64A^2 B^3 B''' - 104AB^2 (AB' + A'B) B'' \right. \\ & + 4ABB'^2 (11AB' + 12A'B) - 4(6AA'' - 13A'^2) B^3 B' + 16AA''' B^4 - 104AA'' B^4 + 112 \frac{A'^3}{A} B^4 \left. \right] r^3 \\ & \left. + \left[20AB^2 B' (AB' + 2A'B) + 16AA'' B^4 - 28A'^2 A'' B^4 \right] r^2 + 32A^2 B^3 B'r + 16A^2 B^4 (A^2 - 1) \right\} / (48A^4 B^4 r^4), \end{aligned} \quad (20)$$

respectively. The term $W^\theta{}_\theta$ may be given by Eq. (17), or simply using the trace identity, i.e., $W^\theta{}_\theta = -(W^t{}_t + W^r{}_r)/2$, through Eqs. (19) and (20).

B. Energy conditions

In this work we are interested in deducing exact solutions of traversable wormholes in conformal Weyl gravity and, therefore, a fundamental point is the energy condition violations. However, a subtle issue needs to be pointed out in this respect. Note that the energy conditions arise when one refers back to the Raychaudhuri equation for the expansion where a term $R_{\mu\nu} k^\mu k^\nu$ appears, and k^μ is a null vector. The positivity of this

quantity ensures that geodesic congruences focus within a finite value of the parameter labelling points on the geodesics. However, in general relativity, through the Einstein field equation one can write the above condition in terms of the stress energy tensor $T_{\mu\nu}$, and consequently one ends up with the null energy condition given by $T_{\mu\nu} k^\mu k^\nu \geq 0$. In any other theory of gravity, one would require to know how one can replace $R_{\mu\nu}$ using the corresponding field equations and hence using matter stresses. In particular, in a theory where we still have an Einstein-Hilbert term, the task of evaluating $R_{\mu\nu} k^\mu k^\nu$ is trivial. However, in the conformal Weyl gravity under consideration, things are not so straightforward.

To this effect, one may rewrite the gravitational field equation (10) in terms of the Einstein tensor, in an anal-

ogous form to the Einstein field equation, given by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{4\alpha}T_{\mu\nu}^{\text{eff}}, \quad (21)$$

where the effective stress energy tensor is given by $T_{\mu\nu}^{\text{eff}} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(W)}$. Note that this relationship differs fundamentally from the Einstein field equation, as one is considering a dimensionless gravitational coupling constant α , contrary to the Newtonian gravitational constant G . Nevertheless, the gravitational field equation written in this form proves extremely useful in deducing a definition of the null energy condition, in terms of the effective stress energy tensor, from the Raychaudhuri expansion term $R_{\mu\nu}k^\mu k^\nu$.

The first term, i.e., $T_{\mu\nu}^{(m)}$, in the effective stress energy tensor, is defined in terms of the matter stress energy tensor, Eq. (9), and is given by

$$T_{\mu\nu}^{(m)} \equiv \frac{3}{2R}T_{\mu\nu}, \quad (22)$$

where R is the curvature scalar.

The second term $T_{\mu\nu}^{(W)}$ may be denoted as the curvature Weyl stress energy tensor, and is provided by

$$T_{\mu\nu}^{(W)} \equiv -\frac{6\alpha}{R}\overline{W}_{\mu\nu}, \quad (23)$$

with the tensor $\overline{W}_{\mu\nu}$ defined as

$$\begin{aligned} \overline{W}_{\mu\nu} = & -\frac{1}{6}g_{\mu\nu}R^{;\beta}_{;\beta} + R_{\mu\nu}^{;\beta}_{;\beta} - R_{\mu}^{\beta}{}_{;\nu\beta} - R_{\nu}^{\beta}{}_{;\mu\beta} \\ & -2R_{\mu\beta}R_{\nu}^{\beta} + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{3}R_{;\mu\nu} + \frac{1}{6}g_{\mu\nu}R^2. \end{aligned} \quad (24)$$

Note that the gravitational field equation (21) imposes interesting conservation equations. Through the Bianchi identities, $G^{\mu\nu}{}_{;\nu} = 0$ and the conservation of the stress energy tensor $T^{\mu\nu}{}_{;\nu} = 0$, which can also be verified from the diffeomorphism invariance of the matter part of the action, one verifies the following conservation law

$$T^{(W)\mu\nu}{}_{;\nu} = \frac{3}{2R^2}T^{\mu\nu}R_{;\nu}. \quad (25)$$

Now the positivity condition, $R_{\mu\nu}k^\mu k^\nu \geq 0$, in the Raychaudhuri equation provides the following form for the null energy condition $T_{\mu\nu}^{\text{eff}}k^\mu k^\nu \geq 0$, through the modified gravitational field equation (21). For this case, in principle, one may impose that the matter stress energy tensor satisfies the energy conditions and the respective violations arise from the Weyl curvature term $T_{\mu\nu}^{(W)}$, in analogy to the case carried out in Ref. [20]. Although this analysis is an interesting avenue to study, we consider an alternative approach which is described below.

Another approach to the energy conditions considers in taking the condition $T_{\mu\nu}k^\mu k^\nu \geq 0$ at face value. Note that this is useful as using local Lorentz transformations it is possible to show that the above condition implies

that the energy density is positive in all local frames of reference. However, if the theory of gravity is chosen to be non-Einsteinian, then the assumption of the above condition does not necessarily imply focusing of geodesics. The focusing criterion is different and will follow from the nature of $R_{\mu\nu}k^\mu k^\nu$. In the next section, we consider this latter approach to the energy conditions, which provides interesting results.

III. TRAVERSABLE WORMHOLES IN CONFORMAL WEYL GRAVITY

In this section, we consider the equations of structure for traversable wormholes in conformal Weyl gravity. For this, it is convenient to express the metric in a more familiar form [15, 16], given by

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (26)$$

where $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate, r , denoted as the redshift function and the form function, respectively [15]. The radial coordinate has a range that increases from a minimum value at r_0 , corresponding to the wormhole throat, to ∞ .

To avoid the presence of event horizons, $\Phi(r)$ is imposed to be finite throughout the coordinate range. At the throat r_0 , one has $b(r_0) = r_0$, which implies that $A(r_0) \rightarrow \infty$. A fundamental condition is the flaring-out condition given by $(b'r - b)/b^2 < 0$, which is provided by the mathematics of embedding [15, 16].

In analogy to their general relativistic counterparts, one may consider asymptotically flat spacetimes. However, it is also possible to match the interior wormhole solution to the unique vacuum solution given by

$$B(r) = A^{-1}(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2, \quad (27)$$

where β , γ and k are constants of integration [1, 2]. Note that the general relativistic Schwarzschild solution is parameterized by β . The constant k characterizes a background de Sitter spacetime, although the metric fields (27) in Weyl gravity correspond to a vacuum solution. The integration constant γ measures departures from the respective solution in classical general relativity. Therefore, it is possible to have a cosmology that admits a de Sitter solution without a cosmological constant [27]. This latter term vanishes identically due to the conformal invariance of the theory. Thus, conformal Weyl gravity naturally avoids the theoretical-observational value discrepancy of the cosmological constant.

In the analysis that follows, we consider that the factor that appears in the gravitational field equation be equal to unity, i.e., $4\alpha = 1$, for notational and computational simplicity.

A. Specific case: constant redshift function

A particularly interesting case are the solutions with a constant redshift function, $\Phi' = 0$. Without a loss of generality one may impose $\Phi = 0$, which is equivalent to considering $B = 1$. This specific case simplifies the field equations significantly, and provide particularly intriguing solutions, which differ from their general relativistic counterparts. This is due to the fact that the fourth order gravitational field equation in conformal Weyl gravity differs from the general relativistic Einstein field equation.

The energy density and radial pressure, taking into account Eqs. (19) and (20), reduce to

$$\rho = -\frac{1}{6r^6} \left\{ 2r^2 (b'''r^2 - 2b''r + 2b') \left(1 - \frac{b}{r} \right) + \left[b''r^2 + \frac{5}{2} (b - b'r) \right] (b - b'r) + 2b^2 \right\}, \quad (28)$$

$$p_r = \frac{1}{6r^6} \left\{ 2r^2 (-b''r + 2b') \left(1 - \frac{b}{r} \right) + \frac{b'r}{2} (b - b'r) + \frac{b}{2} (3b + b'r) \right\}, \quad (29)$$

respectively. The NEC is given by $T_{\mu\nu}k^\mu k^\nu \geq 0$, as mentioned in the Introduction, and for a diagonal stress energy tensor takes the form $\rho + p_r \geq 0$. For the present case, the NEC is given by

$$\rho + p_r = -\frac{1}{6r^6} \left\{ 2r^3 (b'''r - b'') \left(1 - \frac{b}{r} \right) + [b''r^2 + 3(b - b'r)] (b - b'r) \right\}. \quad (30)$$

To verify the non-violation of the NEC at the throat, Eq. (30) imposes the following inequality

$$b''r_0 \leq 3(b' - 1), \quad (31)$$

where the flaring-out condition evaluated at the throat has been taken into account, i.e., $b'(r_0) < 1$. We consider next specific choices for the form function.

1. Form function: $b(r) = r_0$

For this case, the stress energy tensor components are given by

$$\rho = -\frac{3r_0^2}{4r^6}, \quad p_r = \frac{r_0^2}{4r^6}. \quad (32)$$

Note that in this simple case, one already obtains a solution that deviates from the general relativistic counterpart, in that the radial pressure is positive at the throat.

Recall that in general relativity the radial pressure is always negative at the throat, implying the necessity of a radial tension to maintain the throat open. In addition to this, we recall that for the specific case of $b(r) = r_0$, the energy density in general relativity is zero, whilst in conformal Weyl gravity it is negative.

The NEC is provided by

$$\rho + p_r = -\frac{r_0^2}{2r^6}, \quad (33)$$

which shows that the NEC is violated throughout the spacetime.

2. Form function: $b(r) = r_0^2/r$

The specific case of $b(r) = r_0^2/r$ corresponds to a negative energy density in general relativity. Equations (28)-(29) provide the following stress energy tensor scenario

$$\rho = \frac{4(3r^2 - 5r_0^2)r_0^2}{3r^8}, \quad p_r = -\frac{4(r^2 - r_0^2)r_0^2}{3r^8}. \quad (34)$$

The energy density negative in the range $r_0 \leq r < \sqrt{5/3}r_0$. This example also differs from its general relativistic counterpart in that the radial pressure is zero at the throat.

The NEC is provided by

$$\rho + p_r = -\frac{8(r^2 - 2r_0^2)r_0^2}{3r^8}. \quad (35)$$

which shows that the NEC is violated for $r_0 \leq r < \sqrt{2}r_0$.

3. Form function: $b(r) = r_0 + \gamma r_0 (1 - r_0/r)$

The specific choice of

$$b(r) = r_0 + \gamma r_0 \left(1 - \frac{r_0}{r} \right), \quad (36)$$

where $0 < \gamma < 1$, is particularly interesting. The stress energy tensor components are somewhat lengthy, so that the respective profile of the energy density, radial pressure and the NEC are depicted in Fig. 1.

It is interesting to note that for this specific case the NEC evaluated at the throat is given by

$$(\rho + p_r)|_{r_0} = -\frac{5\gamma^2 - 8\gamma + 3}{6r_0^4}. \quad (37)$$

This choice does indeed eliminate the need for the violation of the NEC, in the interval $0.6 \leq \gamma < 1$. Note that this is consistent with the general condition given by inequality (31). Nevertheless, the energy density is negative throughout the spacetime, which violates the weak energy condition (WEC). The WEC, $T_{\mu\nu}U^\mu U^\nu \geq 0$, where U^μ is a timelike vector, implies $\rho \geq 0$ and $\rho + p_r \geq 0$. Note that the radial pressure is positive

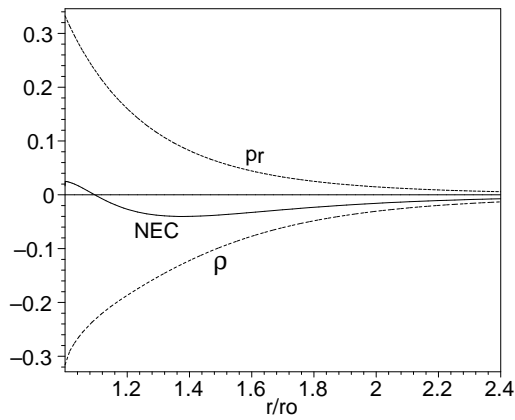


FIG. 1: The energy density, radial pressure and NEC profile for the specific case of $\Phi'(r) = 0$ and $b(r) = r_0 + \gamma r_0(1 - r_0/r)$ for $\gamma = 0.9$. The energy density is negative, the radial pressure positive; and the NEC is satisfied at the throat neighborhood. In particular, at the throat the NEC is satisfied in the range of $0.6 \leq \gamma < 1$. See the text for details.

as depicted in Fig. 1. The specific case of $\gamma = 0.9$ has been used in the figure, which may be considered as a representative for this specific case.

The qualitative behavior of the NEC is depicted in Fig. 2. Note that the NEC is satisfied for high values of γ and low values of r . In particular, the NEC is satisfied for increasing values of r , as γ tends to its limiting value of 1.

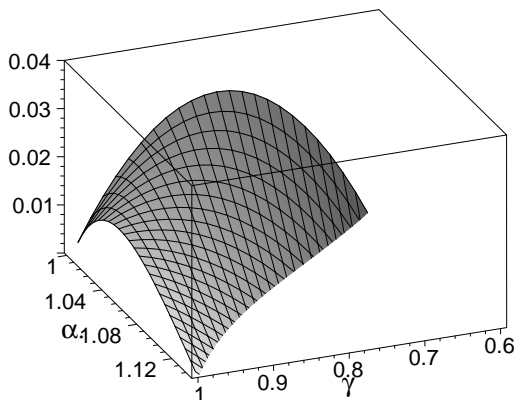


FIG. 2: The NEC profile, with $\rho + p_r \geq 0$, for the specific case of $\Phi'(r) = 0$ and $b(r) = r_0 + \gamma r_0(1 - r_0/r)$. We have defined $\alpha = r/r_0$. The NEC is satisfied at the throat in the range of $0.6 \leq \gamma < 1$. One verifies, qualitatively, that the NEC is satisfied for high values of γ and low values of r , i.e., as r increases, then γ tends to its limiting value of 1.

B. Specific case: $\Phi(r) = r_0/r$

For this case the stress energy tensor components are extremely lengthy, so that they are also depicted in the

respective plots for the specific choices of the form function, considered below.

1. Form function: $b(r) = r_0$

The energy density, radial pressure and NEC are depicted in Fig. 3. Note that the radial pressure is zero at the throat, and then remains negative throughout the coordinate range. The energy density and the NEC are negative in the throat neighborhood.

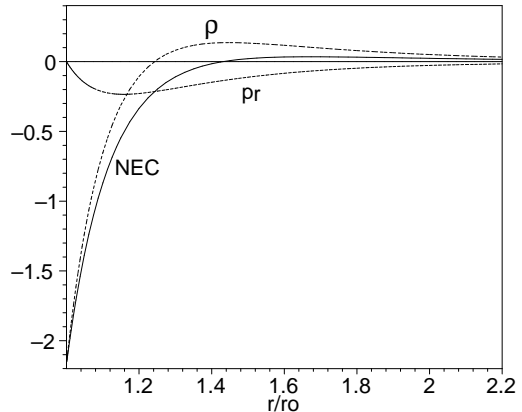


FIG. 3: The energy density, radial pressure and NEC profile for the specific case of $\Phi(r) = r_0/r$ and $b(r) = r_0$. The radial pressure is zero at the throat; the energy density is negative and the NEC is violated in the throat's neighborhood.

2. Form function: $b(r) = r_0^2/r$

The energy density, radial pressure and NEC are depicted in Fig. 4. This choice is qualitatively analogous to the previous case, except that the radial pressure is negative at the throat.

C. Specific case: $\Phi(r) = -r_0/r$ and $b(r) = r_0$

This specific example is a considerable improvement to the solutions considered above. The energy density, radial pressure and NEC profile are depicted in Fig. 5. Note that the pressure is always positive, and the energy density and NEC are also positive in the neighborhood of the throat, thus satisfying all of the energy conditions.

The profile for the specific case of $b(r) = r_0^2/r$ is qualitatively analogous to this case. One may then match these solutions to the exterior vacuum given by Eq. (27), at a junction surface a_0 , in which the energy conditions are satisfied in the interval $r_0 \leq r \leq a_0$. This shows that one may, in principle, construct a class of traversable

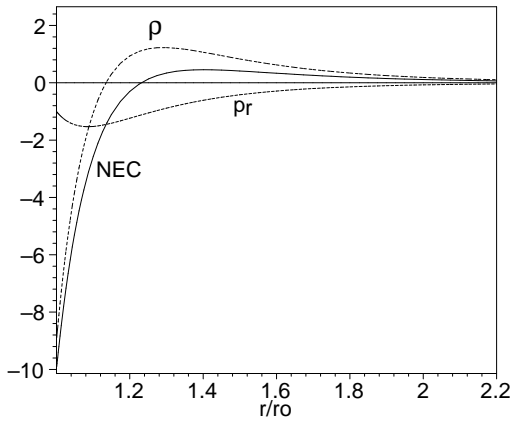


FIG. 4: The energy density, radial pressure and NEC profile for the specific case of $\Phi(r) = r_0/r$ and $b(r) = r_0^2/r$. The radial pressure is negative throughout the spacetime; the energy density is negative and the NEC is violated in the neighborhood of the throat.

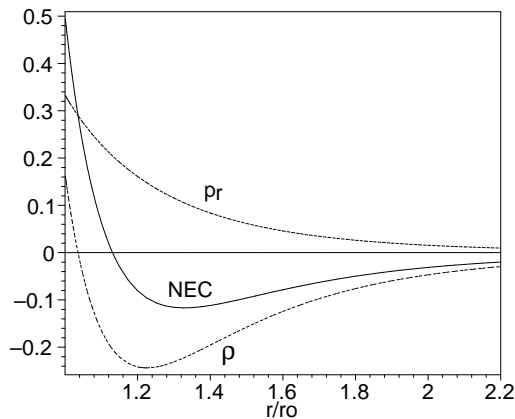


FIG. 5: The energy density, radial pressure and NEC profile for the specific case of $\Phi(r) = -r_0/r$ and $b(r) = r_0$. The radial pressure is positive throughout. The energy density and the NEC are positive in the throat's neighborhood. Consequently, this example shows that one may, in principle, construct a class of traversable wormholes, within the context of conformal Weyl gravity, that satisfies all of the energy conditions, in the vicinity of the throat.

wormholes, within the context of conformal Weyl gravity, that satisfies all of the energy conditions, contrary to their general relativistic counterparts.

IV. CONCLUSION

In general relativity, the null energy condition violation is a fundamental ingredient of static traversable

wormholes. Despite this fact, it was shown that for time-dependent wormhole solutions the null energy condition and the weak energy condition can be avoided in certain regions and for specific intervals of time at the throat [28]. Nevertheless, in certain alternative theories to general relativity, taking into account the modified Einstein field equation, one may impose in principle that the stress energy tensor threading the wormhole satisfies the NEC. However, we emphasize that the latter is necessarily violated by an effective total stress energy tensor. This is the case, for instance, in braneworld wormhole solutions, where the matter confined on the brane satisfies the energy conditions, and it is the local high-energy bulk effects and nonlocal corrections from the Weyl curvature in the bulk that induce a NEC violating signature on the brane [20]. Another particularly interesting example is in the context of the D -dimensional Einstein-Gauss-Bonnet theory of gravitation [17], where it was shown that the weak energy condition can be satisfied depending on the parameters of the theory.

In this work, a general class of wormhole geometries in conformal Weyl gravity was analyzed. In conformal Weyl gravity, as the fourth order gravitational field equations differ radically from the Einstein field equation, one would expect a wider class of solutions. This is indeed the case, in which the stress energy tensor profile differs radically from its general relativistic counterpart, amongst which we may refer to a zero or positive radial pressure at the throat, or at a more fundamental level, the non-violation of the energy conditions in the throat neighborhood, which is in clear contrast to the classical general relativistic static wormhole solutions. Note that as for their general relativistic counterparts, these Weyl variations have far-reaching physical implications, namely apart from being used for interstellar shortcuts, and being multiply-connected spacetimes an absurdly advanced civilization may convert them into time-machines [16, 29, 30], probably implying the violation of causality.

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